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Phugoid Approximation for Conventional Airplanes

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An improved closed-form approximation for phugoid motion in conventional airplanes is presented. Although several closed-form approximations for phugoid motion are currently available and widely used, none of these approximations accurately predict all of the fundamental characteristics of phugoid motion. The new approximation accounts for changes in angle of attack as well as the effects of pitch stability and pitch damping. The total phugoid damping is shown to depend on pitch damping as well as aircraft drag. In addition, this solution points out another important contribution to phugoid damping called phase damping. It is shown that the phase-damping contribution to the real component of the phugoid eigenvalue is always positive and tends to reduce the total phugoid damping. Under certain conditions this phase damping can cause the phugoid mode to become divergent.

Nomenclature

A_w = planform area of the wing
C_D = total drag coefficient
C_{Dp} = parasitic drag coefficient
$C_{D,\alpha}$ = change in drag coefficient with angle of attack
C_L = lift coefficient
$C_{L,\alpha}$ = change in lift coefficient with angle of attack
C_M = pitching moment coefficient
$C_{M,\alpha}$ = change in pitching moment coefficient with angle of attack
$C_{M,\varpi}$ = change in pitching moment coefficient with dimensionless pitching rate
$\bar{c}$ = mean chord length
e = Oswald efficiency factor
F_T = thrust force
g = acceleration of gravity
I_{yy} = pitching moment of inertia in body-fixed coordinates
m = aircraft mass
R_A = aspect ratio
R_d = phugoid pitch-damping ratio
R_g = dimensionless gravitational acceleration
R_M = dimensionless change in pitching moment with axial velocity
$R_{M,\alpha}$ = dimensionless change in pitching moment with angle of attack
$R_{M,\varpi}$ = dimensionless change in pitching moment with pitching rate
R_p = phugoid phase-divergence ratio
R_s = phugoid stability ratio
R_x = dimensionless change in axial force with axial velocity
R_{xa} = complex amplitude
R_{xc} = complex coefficient
R_{xp} = complex phase
$R_{x,\alpha}$ = dimensionless change in axial force with angle of attack
R_z = dimensionless change in normal force with axial velocity
R_{za} = complex amplitude
R_{zc} = complex coefficient
R_{zp} = complex phase
$R_{z,\alpha}$ = dimensionless change in normal force with angle of attack
t = time
V = airspeed
V_{x_b} = axial velocity component in body-fixed coordinates
V_{z_b} = normal velocity component in body-fixed coordinates
V_0 = equilibrium airspeed
x_f = horizontal position in Earth-fixed coordinates
x_T = axial thrust offset in body-fixed coordinates
z_f = vertical position in Earth-fixed coordinates
z_T = normal thrust offset in body-fixed coordinates
α = angle of attack
α_T = thrust angle
Δ = deviation from equilibrium
ζ = dimensionless vertical position
θ = Euler elevation angle or pitch attitude
λ = eigenvalue
λ_i = imaginary part of eigenvalue
λ_r = real part of eigenvalue
ξ = dimensionless horizontal position
ρ = air density
τ = dimensionless time
υ = dimensionless forward velocity
ω_{y_b} = pitching rate in body-fixed coordinates
ϖ = dimensionless pitching rate
ϖ_n = dimensionless undamped natural frequency

Introduction

THE low-frequency oscillations in altitude and airspeed that develop when an airplane is disturbed from equilibrium flight are referred to as phugoid motion. The motion is a slow oscillatory interchange between kinetic and potential energy that occurs when a statically stable aircraft attempts to reestablish the equilibrium balance between lift, weight, thrust, and drag. This periodic motion has been studied for nearly 100 years and is well understood. Lanchester[1] published the first description of phugoid motion, and in this work he presents the first elementary theory of dynamic aircraft stability. Shortly thereafter, a more rigorous mathematical treatment of aircraft motion and dynamic flight stability was developed and published by Bryan.[2] Together, the work of Lanchester and Bryan laid the foundation for the study of dynamic flight stability in general and phugoid motion in particular. In the following decades, much was published on theoretical and experimental investigations of dynamic flight stability and the application of this work to aircraft handling characteristics and design. Perkins[3] has presented a detailed review of this early work. For the past 50 years, the theoretical analysis of phugoid motion and its application to aircraft design has been a topic treated in virtually all engineering textbooks dealing with aircraft stability and control (see Refs. 4–14).

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One approach to the study of longitudinal aircraft dynamics involves solving the full nonlinear equations of motion.[15] This system of nonlinear equations is quite complex. Whereas explicit analytic solutions are rare, such solutions have been obtained for one degree of freedom (for example, see Painlevè[16] and Campos et al.[17]). Nonlinear longitudinal dynamics with three degrees of freedom is typically treated using the method of bifurcation together with numerical methods. Campos[18] gives a good review of the work on nonlinear aircraft dynamics.

A more common approach to longitudinal aircraft dynamics starts with the linearized equations of motion that were first developed by Bryan.[2] Linearized phugoid motion is characterized by the frequency, the damping rate, and the relative amplitudes and phase shifts for the oscillations in airspeed, angle of attack, pitching rate, and altitude. Once the aerodynamic stability and damping derivatives have been determined from wind-tunnel tests or other means, the free flight phugoid characteristics for an aircraft can readily be evaluated. This can be done by numerically determining the eigenvalues and eigenvectors associated with the linearized equations of motion (for example, see Etkin and Reid[13]). However, the eigenvalues and eigenvectors for phugoid motion depend on many aircraft design and operating parameters and the nature of this dependence is not easily observable from a numerical solution. For this reason, a closed-form solution that accurately describes the essential features of phugoid motion is desirable. In addition, closed-form solutions have always been useful for the optimization of aircraft control systems (see Ashkenas and McRuer[19]).

Lanchester[1] developed the first closed-form approximation for phugoid motion. In his original solution, Lanchester assumed no change in angle of attack and no change in the net axial force. With these assumptions, Lanchester obtained an approximation for the phugoid frequency. However, this approximation predicts completely undamped sinusoidal motion and gives no information about the phugoid damping. A well-known modification of Lanchester's solution, which does include an approximation for the phugoid damping, has been widely used. In this approximation, Lanchester's original assumption of no change in angle of attack is retained but the assumption of no change in axial force is dropped. Several variations from this constant angle of attack approximation have been used.[20-23] A somewhat general variation of this approximation was recently presented by Bloy.[24] In this approximation, which accounts for the effects of thrust offset, the assumption of no change in angle of attack is relaxed. Instead, the approximation is based on neglecting all time derivatives in the pitching moment equation. The more commonly used constant angle of attack approximation is obtained, as a special case, from Bloy's solution when the thrust vector is aligned with the center of gravity.

Although several variations of a closed-form phugoid approximation have been used, none of these accurately predict many of the fundamental characteristics of phugoid motion. The current paper presents an improved closed-form approximation that more accurately describes both the period and the damping for phugoid motion. This new solution accounts for changes in angle of attack and accurately predicts the effects of pitch stability and pitch damping. In addition, the solution points out another contribution to phugoid damping that the author has called phase damping. Under certain conditions this phase damping can cause the phugoid mode to become divergent.

Current Phugoid Approximations

To obtain the phugoid eigenvalues and eigenvectors, whether numerically or analytically, we start with the linearized longitudinal equations of aircraft motion. The development of these equations can be found in any undergraduate textbook dealing with aircraft dynamics (for example, Etkin and Reid[13]). The eigenvalues and eigenvectors are obtained from the homogeneous equations, with all control inputs set to zero. For phugoid motion, the oscillations in angle of attack are small and of low frequency. Thus, for this motion, we neglect the force and moment derivatives with respect to the rate of change of angle of attack. Similarly, we also neglect the change in axial and normal force with respect to pitching rate. If we also restrict the analysis to deviations about level flight, the familiar longitudinal equations of motion can be written in dimensionless form as

$$\begin{Bmatrix} \Delta\hat{\upsilon} \\ \Delta\hat{\alpha} \\ \Delta\hat{\varpi} \\ \Delta\hat{\xi} \\ \Delta\hat{\zeta} \\ \Delta\hat{\theta} \end{Bmatrix} = \begin{bmatrix} R_x & R_{x,\alpha} & 0 & 0 & 0 & -R_g \\ R_z & R_{z,\alpha} & 1 & 0 & 0 & 0 \\ R_M & R_{M,\alpha} & R_{M,\varpi} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} \tag{1}$$

As is the usual convention, the characteristic length is taken to be one-half the mean chord length, and the characteristic velocity is taken to be the equilibrium airspeed. Thus,

$$\Delta\upsilon \equiv \Delta V_{x_b}/V_0, \qquad \Delta\alpha \cong \Delta V_{z_b}/V_0, \qquad \Delta\varpi \equiv \Delta\omega_{y_b}\bar{c}/2V_0$$
$$\Delta\xi \equiv 2\Delta x_f/\bar{c}, \qquad \Delta\zeta \equiv 2\Delta z_f/\bar{c} \tag{2}$$

where the Δ indicates a deviation from equilibrium and the notation used on the left-hand side of Eq. (1) indicates differentiation with respect to dimensionless time,

$$\hat{f} \equiv \frac{\bar{c}}{2V_0}\frac{\partial f}{\partial t} \equiv \frac{\partial f}{\partial \tau}, \qquad \tau \equiv \frac{2V_0 t}{\bar{c}} \tag{3}$$

The dimensionless coefficients on the right-hand side of Eq. (1) are all evaluated at the equilibrium flight condition and are defined as

$$R_x \equiv \frac{\rho A_w \bar{c}}{2m}\left[-C_D + \frac{\cos(\alpha_T)}{\rho A_w V_0}\frac{\partial F_T}{\partial V}\right]$$

$$R_z \equiv \frac{\rho A_w \bar{c}}{2m}\left[-C_L - \frac{\sin(\alpha_T)}{\rho V_0 A_w}\frac{\partial F_T}{\partial V}\right]$$

$$R_M \equiv \frac{\rho A_w \bar{c}^3}{4I_{yy_b}}\left(\frac{-F_T}{\frac{1}{2}\rho V_0^2 A_w} + \frac{1}{\rho V_0 A_w}\frac{\partial F_T}{\partial V}\right) \times \frac{[z_T\cos(\alpha_T) + x_T\sin(\alpha_T)]}{\bar{c}}$$

$$R_g \equiv \frac{g\bar{c}}{2V_0^2}$$

$$R_{x,\alpha} \equiv \frac{\rho A_w \bar{c}}{4m}\left(C_L - \frac{\partial C_D}{\partial \alpha}\right) \equiv \frac{\rho A_w \bar{c}}{4m}C_{x,\alpha}$$

$$R_{z,\alpha} \equiv \frac{\rho A_w \bar{c}}{4m}\left(-\frac{\partial C_L}{\partial \alpha} - C_D\right) \equiv \frac{\rho A_w \bar{c}}{4m}C_{z,\alpha}$$

$$R_{M,\alpha} \equiv \frac{\rho A_w \bar{c}^3}{8I_{yy}}\frac{\partial C_M}{\partial \alpha} \equiv \frac{\rho A_w \bar{c}^3}{8I_{yy}}C_{M,\alpha}$$

$$R_{M,\varpi} \equiv \frac{\rho A_w \bar{c}^3}{8I_{yy}}\frac{\partial C_M}{\partial \varpi} \equiv \frac{\rho A_w \bar{c}^3}{8I_{yy}}C_{M,\varpi} \tag{4}$$

The best known and most widely used phugoid approximation is obtained from Eq. (1) by setting the deviation in angle of attack equal to zero and ignoring the pitching moment equation. A more general variation of this approximation, due to Bloy,[24] is obtained from Eq. (1) by simply neglecting all time derivatives in the pitching moment equation. Most of the commonly used variations of the phugoid approximation can be obtained as special cases of Bloy's solution. With this approximation, the eigenproblem associated with Eq. (1) becomes

$$\begin{bmatrix} (R_x-\lambda) & R_{x,\alpha} & 0 & 0 & 0 & -R_g \\ R_z & (R_{z,\alpha}-\lambda) & 1 & 0 & 0 & 0 \\ R_M & R_{M,\alpha} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & 0 & 0 & -\lambda \end{bmatrix} \times \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5}$$

The nontrivial eigenvalues obtained from Eq. (5) are

$$\lambda = \left[\frac{R_x}{2} - \frac{R_M(R_{x,\alpha}-R_g)}{2R_{M,\alpha}}\right] \pm i\sqrt{R_g\left(-R_z + \frac{R_M}{R_{M,\alpha}}R_{z,\alpha}\right) - \left[\frac{R_x}{2} - \frac{R_M(R_{x,\alpha}-R_g)}{2R_{M,\alpha}}\right]^2} \tag{6}$$

and the associated eigenvectors are given by

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} \lambda \\ -(R_M/R_{M,\alpha})\lambda \\ [-R_z + (R_{z,\alpha}-\lambda)(R_M/R_{M,\alpha})]\lambda \\ [R_z - (R_{z,\alpha}R_M/R_{M,\alpha})]/\lambda \\ -R_z + (R_{z,\alpha}-\lambda)(R_M/R_{M,\alpha}) \end{Bmatrix} \Delta\xi \tag{7}$$

If we also assume that the thrust is constant, aligned with the center of gravity, and in the direction of flight, the equilibrium lift force must equal the weight and the equilibrium pitching moment must vanish. For this common configuration, from Eq. (4) we can write

$$R_z = -\frac{\rho A_w \bar{c}}{2m}C_L = -\frac{\rho A_w \bar{c}}{2m}\left(\frac{mg}{\frac{1}{2}\rho V_0^2 A_w}\right) = -\frac{g\bar{c}}{V_0^2} = -2R_g \tag{8}$$

$$R_x = -(\rho A_w \bar{c}/2m)C_D = -(\rho A_w \bar{c}/2m)C_L(C_D/C_L) = -2R_g(C_D/C_L) \tag{9}$$

and

$$R_M = 0 \tag{10}$$

With these additional restrictions, the approximate phugoid eigenvalues and eigenvectors from Eqs. (6) and (7) reduce to the well-known form

$$\lambda = R_g\left[-(C_D/C_L) \pm i\sqrt{2-(C_D/C_L)^2}\right] = (g\bar{c}/2V_0^2)\left[-(C_D/C_L) \pm i\sqrt{2-(C_D/C_L)^2}\right] \tag{11}$$

and

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} \lambda \\ 0 \\ 2R_g\lambda \\ -2R_g/\lambda \\ 2R_g \end{Bmatrix} \Delta\xi \tag{12}$$

Equation (11) predicts the same undamped natural frequency as Lanchester's original approximation.[1] This very simple result gives an undamped phugoid frequency that depends only on airspeed and not at all on the airplane or its altitude. The phugoid damping as predicted by Eq. (11) is also quite simple, depending only on the airspeed and the lift-to-drag ratio for the airplane. Whereas this approximate closed-form solution has been widely used for many years, it is not particularly accurate and does not capture many of the fundamental characteristics of phugoid motion. Because this approximation ignores all angular momentum terms and results in no change in angle of attack, it includes only the effects of lift and drag and does not account for pitch stability, pitch damping, or changes in angle of attack.

For a typical general aviation aircraft the approximate phugoid eigenvalues, computed from Eq. (11), are accurate to within about 20–40%. However, for high-performance aircraft, the result predicted by Eq. (11) is much worse. In fact, under certain conditions, this approximation does not even predict the correct sign for the phugoid damping. Because the gravitational acceleration, the airspeed, the lift coefficient, and the drag coefficient are always positive, Eq. (11) will always predict a convergent phugoid mode. In reality, however, modern airplanes can exhibit a divergent phugoid mode at certain airspeeds.

The approximation given by Eqs. (6) and (7) does not accurately describe phugoid motion under many conditions. Bairstow[20] developed another closed-form approximation that gives a much better result for the phugoid frequency. However, this approximation was never widely used, because in most cases the damping predicted by Bairstow's approximation is less accurate than that predicted by Eq. (11).

Improved Phugoid Approximation

Phugoid motion is always underdamped. Thus, the eigenvalues describing the phugoid mode form a complex pair, $\lambda = \lambda_r \pm i\lambda_i$, where the imaginary part specifies the frequency and the real part specifies the damping. With this recognition, the eigenproblem associated with Eq. (1) could be written as

$$\begin{Bmatrix} (\lambda_r \pm i\lambda_i)\Delta\upsilon \\ (\lambda_r \pm i\lambda_i)\Delta\alpha \\ (\lambda_r \pm i\lambda_i)\Delta\varpi \\ (\lambda_r \pm i\lambda_i)\Delta\xi \\ (\lambda_r \pm i\lambda_i)\Delta\zeta \\ (\lambda_r \pm i\lambda_i)\Delta\theta \end{Bmatrix} = \begin{bmatrix} R_x & R_{x,\alpha} & 0 & 0 & 0 & -R_g \\ R_z & R_{z,\alpha} & 1 & 0 & 0 & 0 \\ R_M & R_{M,\alpha} & R_{M,\varpi} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} \tag{13}$$

Phugoid motion is lightly damped. Thus, the real part of the phugoid eigenvalue pair is very small. Furthermore, whereas both the angle of attack and the pitching rate change significantly with time, the amplitudes for these changes are also very small. Thus, the products $\lambda_r\Delta\alpha$ and $\lambda_r\Delta\varpi$ are extremely small and can be ignored. This means that we can approximate the variation in angle of attack and pitching rate as undamped sinusoidal motion, $(\lambda_r \pm i\lambda_i)\Delta\alpha \cong \pm i\varpi_n\Delta\alpha$ and $(\lambda_r \pm i\lambda_i)\Delta\varpi \cong \pm i\varpi_n\Delta\varpi$, where ϖ_n is the dimensionless undamped natural frequency. For the sake of simplicity, we shall continue to restrict the analysis to deviations about level flight and assume that the thrust vector is aligned with the center of gravity. With these approximations, the longitudinal eigenproblem in Eq. (13) becomes

$$\begin{Bmatrix} \lambda\Delta\upsilon \\ \pm i\varpi_n\Delta\alpha \\ \pm i\varpi_n\Delta\varpi \\ \lambda\Delta\xi \\ \lambda\Delta\zeta \\ \lambda\Delta\theta \end{Bmatrix} = \begin{bmatrix} R_x & R_{x,\alpha} & 0 & 0 & 0 & -R_g \\ R_z & R_{z,\alpha} & 1 & 0 & 0 & 0 \\ 0 & R_{M,\alpha} & R_{M,\varpi} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} \tag{14}$$

The second and third equations in Eq. (14) can be combined to relate the variation in angle of attack and pitching rate to the variation in forward velocity. This gives

$$\begin{Bmatrix} \lambda\Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \lambda\Delta\xi \\ \lambda\Delta\zeta \\ \lambda\Delta\theta \end{Bmatrix} = \begin{bmatrix} R_x & R_{x,\alpha} & 0 & 0 & 0 & -R_g \\ R_{xc} & 0 & 0 & 0 & 0 & 0 \\ R_{zc} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} \tag{15}$$

where the complex coefficients R_{xc} and R_{zc} are defined as

$$R_{xc} \equiv \frac{R_z(R_{M,\varpi} \mp i\varpi_n)}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi} + \varpi_n^2 \pm i\varpi_n(R_{z,\alpha} + R_{M,\varpi})} \tag{16}$$

and

$$R_{zc} \equiv \frac{-R_z R_{M,\alpha}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi} + \varpi_n^2 \pm i\varpi_n(R_{z,\alpha} + R_{M,\varpi})} \tag{17}$$

The second and third equations in Eq. (15) can now be used to eliminate the angle of attack and the pitching rate from the first and last of these equations. Thus, Eq. (15) can be rewritten as

$$\begin{bmatrix} (R_x + R_{x,\alpha}R_{xc} - \lambda) & 0 & 0 & 0 & 0 & -R_g \\ R_{xc} & -1 & 0 & 0 & 0 & 0 \\ R_{zc} & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & 0 & -\lambda & -1 \\ R_{zc} & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \times \begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{18}$$

The first and the last of the equations in Eq. (18) contain only the dimensionless forward velocity deviation $\Delta\upsilon$ and the deviation in the Euler elevation angle $\Delta\theta$. These two equations can be separated from the other four equations, and this system can be rearranged to give the second-order eigenproblem,

$$\begin{bmatrix} (R_x + R_{x,\alpha}R_{xc} - \lambda) & -R_g \\ R_{zc} & -\lambda \end{bmatrix} \begin{Bmatrix} \Delta\upsilon \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = \begin{Bmatrix} \lambda \\ R_{xc}\lambda \\ R_{zc}\lambda \\ R_{xc} - R_{zc}/\lambda \\ R_{zc} \end{Bmatrix} \Delta\xi \tag{20}$$

Equation (19) can be used to solve for the phugoid eigenvalues and the resulting complex eigenvalues can be used in Eq. (20) to obtain the eigenvectors.

The characteristic equation associated with Eq. (19) is

$$\lambda^2 - (R_x + R_{x,\alpha}R_{xc})\lambda + R_g R_{zc} = 0 \tag{21}$$

and the eigenvalues are the roots of this quadratic equation,

$$\lambda = \frac{R_x + R_{x,\alpha}R_{xc}}{2} \pm \sqrt{\left(\frac{R_x + R_{x,\alpha}R_{xc}}{2}\right)^2 - R_g R_{zc}} \tag{22}$$

Because the phugoid frequency is very low, we can neglect terms in Eqs. (16) and (17) that contain the phugoid frequency raised to any power greater than one. Thus, these complex coefficients can be approximated as

$$R_{xc} \cong \frac{R_z R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}\left(1 \mp i\frac{\varpi_n}{R_{M,\varpi}}\right) \times \left[1 \mp i\frac{\varpi_n(R_{z,\alpha} + R_{M,\varpi})}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}\right] \cong R_{xa}(1 \mp i\varpi_n R_{xp}) \tag{23}$$

and

$$R_{zc} \cong \frac{-R_z R_{M,\alpha}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}\left[1 \mp i\frac{\varpi_n(R_{z,\alpha} + R_{M,\varpi})}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}\right] \cong R_{za}(1 \mp i\varpi_n R_{zp}) \tag{24}$$

where R_{xa}, R_{za}, R_{xp}, and R_{zp} are defined as

$$R_{xa} \equiv \frac{R_z R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}} \tag{25}$$

$$R_{za} \equiv \frac{-R_z R_{M,\alpha}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}} \tag{26}$$

$$R_{xp} \equiv \frac{R_{M,\alpha} + R_{M,\varpi}^2}{R_{M,\varpi}(R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi})} \tag{27}$$

and

$$R_{zp} \equiv \frac{R_{z,\alpha} + R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}} \tag{28}$$

Using Eqs. (23) and (24) in Eq. (22), the phugoid eigenvalues are approximated as

$$\lambda \cong \frac{R_x + R_{x,\alpha}R_{xa}(1 \mp i\varpi_n R_{xp})}{2} \pm \sqrt{\left[\frac{R_x + R_{x,\alpha}R_{xa}(1 \mp i\varpi_n R_{xp})}{2}\right]^2 - R_g R_{za}(1 \mp i\varpi_n R_{zp})} \tag{29}$$

The imaginary components of the complex coefficients R_{xc} and R_{zc} are very small. This very small phase shift transfers a small fraction of the imaginary part of the eigenvalue to the real part and transfers a small fraction of the real part to the imaginary part. Because phugoid motion is so lightly damped, the small fraction of the real part that is phase shifted to the imaginary part can be neglected. However, when a very small fraction of the much larger imaginary part is phase shifted to the real part, it becomes significant.

Thus, ignoring the phase shift in R_{xc}, Eq. (29) can be written

$$\lambda \cong \frac{R_x + R_{x,\alpha}R_{xa}}{2} \pm i\sqrt{R_g R_{za}}\sqrt{1 \mp i\varpi_n R_{zp} - \frac{(R_x + R_{x,\alpha}R_{xa})^2}{4R_g R_{za}}} \tag{30}$$

Because both the phase shift and the square of the damping term are very small, we can further approximate Eq. (30) as

$$\lambda \cong \frac{R_x + R_{x,\alpha}R_{xa}}{2} \pm i\sqrt{R_g R_{za}}\left[1 \mp i\frac{\varpi_n R_{zp}}{2} - \frac{(R_x + R_{x,\alpha}R_{xa})^2}{8R_g R_{za}}\right] \tag{31}$$

which can be rearranged to give

$$\lambda \cong [(R_x + R_{x,\alpha}R_{xa})/2] + (\varpi_n R_{zp}\sqrt{R_g R_{za}}/2) \pm i\sqrt{R_g R_{za} - [(R_x + R_{x,\alpha}R_{xa})/2]^2} \tag{32}$$

From Eq. (32) it is clearly seen that there is a component of phugoid damping that is related to the phugoid frequency. Here this will be called phase damping because it results from the phase shift between angle of attack, pitching rate, and forward velocity. As will be shown, the phase-damping contribution to the real component of the eigenvalue is always positive and tends to reduce the total phugoid damping. In fact, under certain circumstances, the phase damping can result in a large enough positive contribution to make the phugoid motion divergent.

The dimensionless undamped natural frequency ϖ_n has to this point been unknown. However, this natural frequency is simply the magnitude of the imaginary part of the phugoid eigenvalues with all damping terms set to zero. Thus, ϖ_n is easily evaluated from Eq. (32) by setting both $(R_x + R_{x,\alpha} R_{xa})/2$ and $[\varpi_n R_{zp} \sqrt{(R_g R_{za})}]/2$ to zero. This gives

$$\varpi_n = \sqrt{R_g R_{za}} \tag{33}$$

By the use of Eq. (33) in Eq. (32), the dimensionless phugoid eigenvalues can be expressed as

$$\lambda \cong (R_z/2)\Big\{[(R_x/R_z) + R_d - R_p] \pm i\sqrt{-(4R_g/R_z)R_s - [(R_x/R_z) + R_d]^2}\Big\} \tag{34}$$

where we define the phugoid stability ratio,

$$R_s \equiv -\frac{R_{za}}{R_z} = \frac{R_{M,\alpha}}{R_{M,\alpha} - R_{z,\alpha} R_{M,\varpi}} \tag{35}$$

the phugoid pitch-damping ratio,

$$R_d \equiv \frac{R_{x,\alpha} R_{xa}}{R_z} = \frac{R_{x,\alpha} R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha} R_{M,\varpi}} \tag{36}$$

and the phugoid phase-divergence ratio,

$$R_p \equiv -\frac{R_{zp} R_g R_{za}}{R_z} = R_g R_s \left(\frac{R_{z,\alpha} + R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha} R_{M,\varpi}} \right) \tag{37}$$

If the thrust is independent of airspeed and aligned with the direction of flight, this approximation can be somewhat simplified by invoking Eqs. (8) and (9). Using these equations in Eq. (34), the dimensionless phugoid eigenvalues can be written as

$$\lambda \cong (g\bar{c}/2V_0^2)\Big\{[-(C_D/C_L) - R_d + R_p] \pm i\sqrt{2R_s - [(C_D/C_L) + R_d]^2}\Big\} \tag{38}$$

Results

The phugoid frequency predicted from Eq. (34) or Eq. (38) depends on the pitch stability of the airplane. The phugoid damping predicted from this approximation is a function of the pitch-damping derivative and the phugoid frequency, as well as the drag. The eigenvalues and eigenvectors predicted by using Eq. (34) or Eq. (38) with Eq. (20) are greatly improved over those predicted from Eqs. (6) and (7). For example, consider a typical general aviation airplane having

$$V_0 = 54.864 \text{ m/s}, \quad \bar{c} = 1.709 \text{ m}, \quad C_L = 0.393$$

$$C_D = 0.050, \quad C_{D,\alpha} = 0.35, \quad C_{L,\alpha} = 4.40$$

$$C_{M,\alpha} = -0.68, \quad C_{M,\varpi} = -9.95, \quad R_x = -0.00071$$

$$R_z = -0.00556, \quad R_M = 0.0, \quad R_g = 0.00278$$

$$R_{x,\alpha} = 0.00030, \quad R_{z,\alpha} = -0.03151$$

$$R_{M,\alpha} = -0.00220, \quad R_{M,\varpi} = -0.03212$$

By the use of these values, the exact solution for the phugoid eigenvalues and eigenvectors obtained numerically from Eq. (1) gives

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = C_{1,2} \begin{Bmatrix} 0.002081 \pm 0.000370i \\ -0.000115 \mp 0.000025i \\ 0.000008 \pm 0.000001i \\ 0.062661 \mp 0.646781i \\ 0.760090 \\ 0.000081 \mp 0.002489i \end{Bmatrix} \times \exp[(-0.000258 \pm 0.003242i)\tau]$$

The approximate solution obtained from Eqs. (20) and (38) results in

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = C_{1,2} \begin{Bmatrix} 0.002077 \pm 0.000370i \\ -0.000115 \mp 0.000025i \\ 0.000008 \pm 0.000001i \\ 0.062894 \mp 0.647399i \\ 0.759545 \\ 0.000081 \mp 0.002481i \end{Bmatrix} \times \exp[(-0.000257 \pm 0.003233i)\tau]$$

whereas the approximate solution obtained from Eqs. (6) and (7) yields

$$\begin{Bmatrix} \Delta\upsilon \\ \Delta\alpha \\ \Delta\varpi \\ \Delta\xi \\ \Delta\zeta \\ \Delta\theta \end{Bmatrix} = C_{1,2} \begin{Bmatrix} 0.002233 \pm 0.000407i \\ 0.000000 \\ 0.000012 \pm 0.000002i \\ 0.051940 \mp 0.575005i \\ 0.816490 \\ 0.000289 \mp 0.003197i \end{Bmatrix} \times \exp[(-0.000354 \pm 0.003916i)\tau]$$

Figures 1–3 show how the phugoid approximation given by Eq. (38) compares with Eq. (11) and the exact solution for a broad range of aerodynamic parameters. Note that, as the pitch-stability derivative approaches infinity, the phugoid stability ratio approaches unity, while both the phugoid pitch-damping ratio and the phase-divergence ratio approach zero. Thus we see that, in the limit of infinite pitch stability, the result given by Eq. (38) reduces exactly to Eq. (11). Thus, Eq. (11) represents an asymptotic solution for infinite pitch stability. This can be seen graphically in Fig. 1. In Fig. 1, all parameters except the pitch-stability derivative $C_{M,\alpha}$ have been held constant at those values given in the earlier example. In Fig. 2, a comparison between Eq. (11), Eq. (38), and the exact solution is shown for a broad range of lift-to-drag ratio. A similar comparison, showing the effect of pitch damping $C_{M,\varpi}$ is shown in Fig. 3.

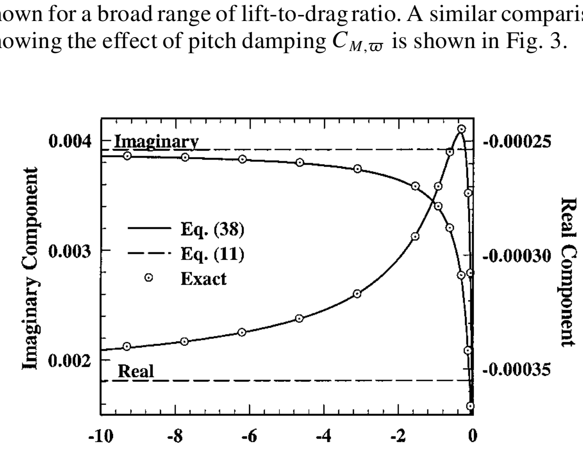

Fig. 1 Effect of pitch stability on the dimensionless phugoid eigenvalues.

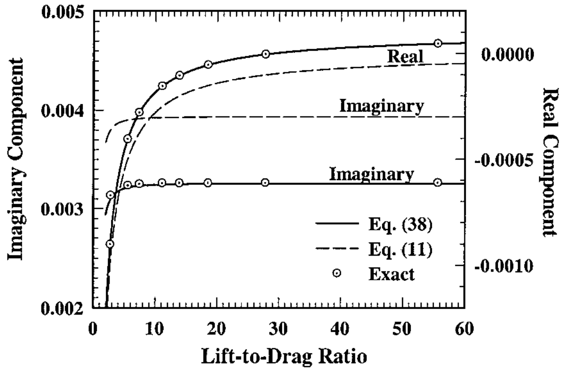

Fig. 2 Effect of lift-to-drag ratio on the dimensionless phugoid eigenvalues.

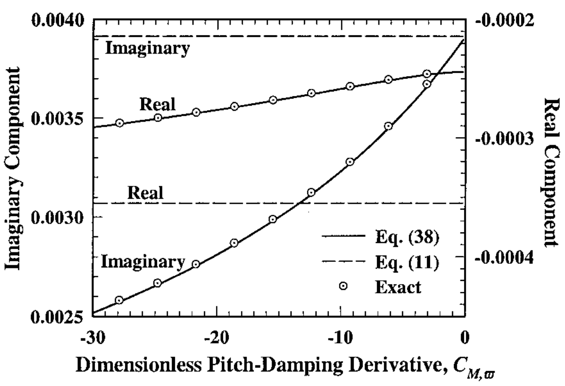

Fig. 3 Effect of pitch damping on the dimensionless phugoid eigenvalues.

Discussion

The present closed-form approximation allows us to see more easily how the aerodynamic coefficients and stability derivatives affect the phugoid motion. Damping has very little effect on the phugoid frequency. Thus, neglecting the damping in Eq. (38) and applying the definition of phugoid stability ratio from Eqs. (35) and (4), the undamped natural frequency for phugoid motion is

$$\omega_n = \frac{2V_0}{\bar{c}}\text{imag}(\lambda) \cong \sqrt{2}\frac{g}{V_0}\sqrt{\frac{R_{M,\alpha}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}}$$

$$= \sqrt{2}\frac{g}{V_0}\sqrt{\frac{(4m/\rho A_w\bar{c})C_{M,\alpha}}{(4m/\rho A_w\bar{c})C_{M,\alpha} - C_{z,\alpha}C_{M,\varpi}}} \tag{39}$$

From Eq. (39), we first notice that the phugoid frequency is inversely proportional to forward speed. In addition, there are four other parameters that affect the phugoid frequency, the dimensionless mass, the change in pitching moment with respect to angle of attack, the change in z force with respect to angle of attack, and the change in pitching moment with respect to pitching rate. For a statically stable aircraft, all of these parameters except the dimensionless mass are negative. Thus, the phugoid stability ratio is always between zero and unity. From this term we see that the phugoid frequency increases with dimensionless aircraft mass and pitch stability, whereas it decreases with lift slope and pitch damping. As the product of dimensionless mass and pitch stability becomes large, compared to the product of lift slope and pitch damping, the phugoid stability ratio approaches unity, and the undamped phugoid frequency predicted from Eq. (38) reduces to that predicted by Eq. (11). Conversely, as the product of pitch stability and dimensionless mass approaches zero, so do the phugoid stability ratio and the phugoid frequency.

From Eq. (38), we see that there are three distinct components to phugoid damping. Here, these are called the drag damping, the pitch damping, and the phase damping. In most cases the drag damping is the largest of these three components. From Eq. (38), we define

$$\text{drag damping} \equiv -(2V_0/\bar{c})\left[(g\bar{c}/2V_0^2)(-C_D/C_L)\right]$$

$$= (g/V_0)(C_D/C_L) \tag{40}$$

Clearly, because the velocity, the drag coefficient, and the lift coefficient are all positive, the drag damping is always positive. The total drag is the sum of the parasitic drag and the induced drag. The parasitic drag coefficient is very nearly constant. The induced drag coefficient is proportional to the lift coefficient squared, and the lift coefficient is inversely proportional to the velocity squared. Thus, we can write

$$\text{drag damping} \cong \frac{g}{V_0}\frac{C_{Dp} + C_L^2/\pi e R_A}{C_L} = \frac{g}{V_0}\left(\frac{C_{Dp}}{C_L} + \frac{C_L}{\pi e R_A}\right)$$

$$= \frac{g}{V_0}\left[\frac{C_{Dp}(\frac{1}{2}\rho V_0^2 A_w)}{mg} + \frac{mg}{\pi e R_A(\frac{1}{2}\rho V_0^2 A_w)}\right] \tag{41}$$

At very high airspeeds, the parasitic drag dominates, the drag-to-lift ratio is proportional to velocity squared, and the phugoid drag damping increases linearly with forward velocity. At very low airspeeds, the induced drag dominates, the drag-to-lift ratio is inversely proportional to velocity squared, and the phugoid drag damping varies inversely with the forward velocity cubed. At some intermediate airspeed, phugoid drag damping will exhibit a minimum.

The phugoid pitch damping is defined from Eqs. (38), (36), and (4) to be

$$\text{pitch damping} \equiv \frac{g}{V_0}R_d = \frac{g}{V_0}\left(\frac{R_{x,\alpha}R_{M,\varpi}}{R_{M,\alpha} - R_{z,\alpha}R_{M,\varpi}}\right)$$

$$= \frac{g}{V_0}\left[\frac{C_{x,\alpha}C_{M,\varpi}}{(4m/\rho A_w\bar{c})C_{m,\alpha} - C_{z,\alpha}C_{M,\varpi}}\right] \tag{42}$$

To determine the sign of the phugoid pitch damping, we note that the dimensionless mass is positive, whereas the pitch-stability derivative, the z-force derivative, and the pitch-damping derivative are all negative. Thus, the phugoid pitch damping will have the same sign as the change in axial force with respect to angle of attack. This axial force derivative can be either positive or negative.

The change in the x-force coefficient with respect to angle of attack can be written in terms of the lift and drag coefficients,

$$C_{x,\alpha} \equiv \frac{\partial}{\partial\alpha}(C_L\sin\alpha - C_D\cos\alpha) = C_L\cos\alpha + C_{L,\alpha}\sin\alpha$$

$$+ C_D\sin\alpha - C_{D,\alpha}\cos\alpha \cong C_L - C_{D,\alpha} \tag{43}$$

The change in drag coefficient with angle of attack can be approximated as

$$C_{D,\alpha} \cong \frac{\partial C_D}{\partial C_L}\frac{\partial C_L}{\partial\alpha} = \frac{\partial}{\partial C_L}\left(C_{Dp} + \frac{C_L^2}{\pi e R_A}\right)C_{L,\alpha} = \frac{2C_{L,\alpha}}{\pi e R_A}C_L$$

$$= \frac{2C_{L,\alpha}}{\pi e R_A}\frac{mg}{\frac{1}{2}\rho V_0^2 A_w} \tag{44}$$

Using Eq. (44) in Eq. (43), we have

$$C_{x,\alpha} \cong C_L\left(1 - \frac{2C_{L,\alpha}}{\pi e R_A}\right) = \frac{mg}{\frac{1}{2}\rho V_0^2 A_w}\left(1 - \frac{2C_{L,\alpha}}{\pi e R_A}\right) \tag{45}$$

By the use of Eq. (45) in Eq. (42), the phugoid pitch damping can be expressed as

$$\text{pitch damping} \cong \frac{mg^2}{\frac{1}{2}\rho V_0^3 A_w}\left(1 - \frac{2C_{L,\alpha}}{\pi e R_A}\right)$$

$$\times\left(\frac{C_{M,\varpi}}{(4m/\rho A_w\bar{c})C_{M,\alpha} - C_{z,\alpha}C_{M,\varpi}}\right) \tag{46}$$

From Eq. (46), we see that phugoid pitch damping is inversely proportional to the cube of the forward velocity; it increases with the pitch-damping derivative; it decreases with increasing pitch stability. We also see that phugoid pitch damping can be either positive or negative. Furthermore, the pitch damping is more negative for planes with low aerodynamic efficiency and more positive for planes with high aerodynamic efficiency. This may seem counterintuitive. However, the total phugoid damping is the sum of the drag damping, the pitch damping, and the phase damping. Lowering the aspect ratio or Oswald efficiency factor will increase the drag damping more than it will decrease the pitch damping. Thus, total phugoid damping will increase with decreasing aerodynamic efficiency, as expected.

Using Eqs. (37) and (4) in Eq. (38), the phugoid phase damping is defined as

$$\text{phase damping} \equiv -\frac{g}{V_0}R_p = -\frac{g}{V_0}R_g R_s\left(\frac{R_{z,\alpha}+R_{M,\varpi}}{R_{M,\alpha}-R_{z,\alpha}R_{M,\varpi}}\right)$$

$$= -\frac{g^2\bar{c}}{2V_0^3}R_s\left[\frac{(8I_{yy}/\rho A_w\bar{c}^3)C_{z,\alpha}+(4m/\rho A_w\bar{c})C_{M,\varpi}}{(4m/\rho A_w\bar{c})C_{M,\alpha}-C_{z,\alpha}C_{M,\varpi}}\right] \tag{47}$$

We have already seen that the phugoid stability ratio can vary from zero to positive unity. The dimensionless mass and moment of inertia are always positive, whereas the pitch-stability derivative, the pitch-damping derivative, and the z-force derivative are all negative. Thus, for a stable aircraft, the phugoid phase damping is always negative, tending to decrease the total phugoid damping. In fact, under certain circumstances, it is possible for this negative phase damping to overpower the drag and pitch damping, making the phugoid motion divergent. Because the phugoid phase-damping is inversely proportional to the forward velocity cubed, this condition is aggravated at low airspeed.

The phugoid approximation given by Eq. (11) predicts that only by increasing drag can we increase phugoid damping. Increasing aircraft drag to improve phugoid damping is obviously not a desirable solution. From the current improved approximation, we see that phugoid damping can, in fact, be increased without increasing aircraft drag. This can be done by either increasing the phugoid pitch damping or by decreasing the magnitude of the negative phugoid phase damping.

Conclusions

An improved closed-form approximation for phugoid motion has been developed. The results show that the phugoid frequency increases with aircraft mass and pitch-stability derivative, whereas it decreases with the lift slope and pitch-damping derivative. As the pitch-stability derivative approaches zero, or the pitch-damping derivative becomes very large, the phugoid frequency approaches zero. The total phugoid damping is shown to depend on pitch damping as well as aircraft drag. It is shown that phugoid pitch damping is inversely proportional to the cube of the forward velocity and increases with the pitch-damping derivative. Phugoid pitch damping decreases with increasing pitch stability and can be either positive or negative. This new closed-form phugoid approximation also points out an additional contribution to phugoid damping that the author has called phase damping. The phugoid phase damping is a function of aircraft mass and moment of inertia, as well as the pitch-damping and pitch-stability derivatives, and is inversely proportional to the forward velocity cubed. For a statically stable aircraft, the phugoid phase damping is always negative, tending to decrease the total phugoid damping, and it is possible for this negative phase damping to render the phugoid motion divergent.

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